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ON THE REALIZATION OF RELIABILITY FUNCTIONS OF PROBABILISTIC COMMUNICATION NETWORKS*

by

Shu-Gar Chan,** Yumin Fu,*** and S. Park Chan****

ABSTRACT

This paper presents some properties and interrelationships of terminal reliability functions of probabilistic communication networks with distinct element reliabilities. Based on topological techniques a synthesis procedure as well as realizability conditions are given. In addition, a uniqueness theorem is proved.

1. *Introduction*

A probabilistic communication network is a network in which each branch is assigned a probability that the specific branch will function properly. This probability is called the reliability of that branch. Thus this network can be represented by a weighted linear graph [1]. The probability of successful communication between a specified pair of stations (terminals) in the network, expressed in terms of branch reliabilities, is called the terminal reliability function.

The analysis of the reliability of probabilistic communication networks has been studied by various investigators [2], [3], [4]. Maxwell [5] has discussed the synthesis of a prescribed terminal reliability function of a two-terminal network in which all branches are assumed to have identical reliabilities. This paper will consider networks with distinct branch reliabilities. The synthesis methods of a single prescribed reliability function as well as two or more given functions corresponding to different pairs of terminals will be discussed. Topological techniques are used here. All terms, if not defined in this paper, may be found in a standard textbook [1].

2. *Basic definitions*

Let the branch reliabilities of a network with e branches be p_1, p_2, ..., p_e. Then p_j is the probability of branch j of the network that this branch will function properly. The complements of these values are denoted by p_1', p_2', ..., p_e'. The probability of transmission between any specified pair of terminals, expressed in terms of branch reliabilities and their complements, is defined as a ***terminal reliability function,*** which is given by

$$q_{i,j} = m_1 p_1' p_2' \dots p_e' + m_2 p_1' p_2' \dots p_{e-1}' p_e + \dots + m_{2^e} p_1 p_2 \dots p_e \qquad (1)$$

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** Department of Electrical Engineering, U.S. Naval Postgraduate School.

*** Department of Electrical Engineering, Southern Methodist University.

**** Department of Electrical Engineering, University of Santa Clara, Santa Clara, Calif., U.S.A.

where m_1, m_2, ..., m_x, ...m_{2^e} are either 1 or 0, depending on the terminal-pair (i, j). That is, m_x is 1 if its associated combination of states of the branches provides a transmission path between the terminals i and j ; otherwise m_x is 0. When a function is expressed as in equation (1), it is said to be in its *canonical form*. Each term in the canonical form is called a *canonical term**. A network with v terminals has $\frac{1}{2}v(v-1)$ terminal reliability functions if the network corresponds to a nonoriented graph.

A *path set* is a set of branches of a communication path (path, in short) between a pair of terminals. A *path product* is the product of the variables associated with the elements in a path set. A canonical term is a *path-product term* (or simply, *path term*) if its unprimed variables are those of a path product and all other variables (in that term) are primed variables. A canonical term is a *secondary path-product term* (or simply, *secondary path term*) if a proper subset of the set of all unprimed variables in the term corresponds to those elements forming a path set.

$P_{i,j}$ is defined to be the collection of all the path sets between terminals i and j. A *circuit set* is a set of elements contained in a circuit. B is defined to be the collection of all possible circuit sets of a graph including the empty set. $B \oplus P_{i,j}$ is a collection of all possible sets, each of which is the *ring sum*** of a set in B and a set in $P_{i,j}$ with respect to terminals i and j.

Thus a set obtained by the ring sum of any set in B and any set in $P_{i,j}$ is in $B \oplus P_{i,j}$, and any set which cannot be obtained by the ring sum of a set in B and a set in $P_{i,j}$ is not in $B \oplus P_{i,j}$.

3. *Properties of terminal reliability functions*

Before a synthesis procedure is developed for terminal reliability functions, some of the properties of, and the interrelationships between these functions, are listed below. The proofs of these statements, (a) through (e), can be found in reference [6].

(a) Let B_k be a set or the ring sum of sets in a collection of e-v+1 linearly independent circuit sets, and let $S_{i,j}$ be a path set in $P_{i,j}$ between vertices (terminals) i and j of a graph. The elements in the set $B_k \oplus S_{i,j}$ are the edges (branches) corresponding to the unprimed variables of a path term or a secondary path term of the terminal reliability function $q_{i,j}$.

(b) Let B_f be a collection of e-v+1 linearly independent circuit sets in B and $S_{i,j}$ be a path set in $P_{i,j}$ between vertices i and j of a graph. Then, among the sets in $B \oplus P_{i,j}$ obtained from B_f and $S_{i,j}$ by the ring sum operation, there is at least one set corresponding to every path term of the terminal reliability function $q_{i,j}$.

(c) The ring sum of a path set $S_{i,j}$ with respect to vertices i and j and a path set $S_{j,k}$ with respect to vertices j and k is either a set in $P_{i,k}$ or the element disjoint union of a set in $P_{i,k}$ and a set in B.

(d) All sets in B of a graph G are obtainable from any $q_{i,j}$ function in G. Also, all collections of possible circuit sets obtained from different $q_{i,j}$ functions in the same graph G are indentical.

(e) Let $P_{f;i,j}$ and $P_{f;j,k}$ be collections of e-v+2 linearly independent sets in $P_{i,j}$ and $P_{j,k}$, respectively, in a graph G. Then all sets in $P_{i,k}$ can be obtained from the sets in $P_{f;i,j}$ and $P_{f;j,k}$ by the ring sum operations.

Using some of the above properties, the following theorems can be established.

* Note that each canonical term includes all the variables, either primed on unprimed.

** The ring sum $S_1 \oplus S_2$ of two sets S_1 and S_2 is the set which includes all elements either in S_1 or in S_2 but not in both.

Theorem 1: One of the terminal reliability functions $q_{i,j}$, $q_{j,k}$ and $q_{i,k}$ of a connected graph G* is a dependent function with respect to the other two.

Proof: If any two of the three functions with respect to three terminal-pairs consisting of three vertices in G are given; e.g., $q_{i,j}$ and $q_{j,k}$ are given, then $P_{i,j}$ and $P_{j,k}$ are known. According to Property (e), $P_{i,k}$ can be obtained. Thus, $q_{i,k}$ can be generated from $q_{i,j}$ and $q_{j,k}$ so that any one of these three functions is a dependent function with respect to the other two.

Theorem 2: If v is the number of vertices in a graph G**, then there are, at most, v-1 independent terminal reliability functions in G.

Proof: Let 1, 2, ..., v be the vertices of the graph G. If we are given $q_{1,2}$, $q_{2,3}$, ..., $q_{v-1,v}$, we can generate all other $\frac{1}{2}$ v(v-1) - (v-1) functions from the given collection of v-1 independent functions according to Theorem 1. Thus, the theorem is true.

4. *Realization of reliability functions*

A terminal reliability function in the canonical form contains all terms, each of which expresses the operating state of every branch of the network. The canonical terms of a terminal reliability function are said to be *conformable* if and only if the function contains all the secondary path terms associated with every path term of the function.

In synthesis, one wishes to construct a graph from a given $q_{i,j}$ function. Since there exists a one-to-one correspondence between the path terms in $q_{i,j}$ and the rows of a path matrix P*** between terminals i and j of the graph G to be realized, P can be readily obtained from $q_{i,j}$ by identifying the path terms of $q_{i,j}$ and the problem is reduced to the realization of G from P. Now, if a column, identified by x_0, is added to P as its $(n+1)^{th}$ column, a new matrix, B_p, results and each row of B_p represents a circuit since any path between i and j, together with the new edge, x_0, connected across i and j of G, forms a circuit. It can be shown**** that a fundamental circuit matrix B_f can be obtained from B_p by the ring-sum operations on the rows of B_p, and the task of finding G from B_f has been solved and is well known [1]. Thus, the following synthesis procedure can be stated without further discussion.****

The Synthesis Procedure. Let $q_{i,j}$ be the given reliability function containing variables $x_1, x_2, \ldots, x_n$ and let the two terminals be i and j.

Step 1: Identify all the path terms of $q_{i,j}$ and form the path matrix P between i and j with $x_1, x_2, \ldots, x_n$ as columns and $p_1, p_2, \ldots, p_k$ as rows where p_r corresponds to the r^{th} path term and k is the total number of path terms of $q_{i,j}$.

Step 2: Add a column of ones to P to form the circuit matrix B_p and

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* To eliminate trivial cases only connected graphs are considered in this paper.

** Including separable graphs as well as non-separble graphs.

*** The path matrix P between vertices i and j of a graph G of v vertices and e edges is defined as the matrix $P = [p_{i,j}]$ with k rows (corresponding to all (K) possible paths between i and j) and e columns (corresponding to all the e edges) where

$p_{i,j} = 1$ if edge j is in path i;
$p_{i,j} = 0$ if edge j is not in path i.

**** A detailed discussion can be found in References [7], Chapter III.

identify the added column by x_o.

Step 3: Find the rank of B_p by reducing it to the normal form using row operations with possible rearrangement of columns. This will produce a fundamental circuit matrix B_f:

$$B_f = \begin{array}{c} \begin{array}{ccccccccc} x'_1 & x'_2 & \cdots & x'_r & x'_{r+1} & \cdots & x'_n & x'_{n+1} \end{array} \\ \left[\begin{array}{c|c} U_r & B_{f_{12}} \end{array} \right] \end{array}$$

where the order $x'_1, x'_2, \ldots, x'_{n+1}$ denotes the new sequence of the columns after B_f is obtained. (Note that the set $x'_1, x'_2, \ldots, x'_r$ constitutes a chord set, and the set $x'_{r+1}, x'_{r+2}, \ldots, x'_{n+1}$ constitutes a treebranch set.)

Step 4: Form the fundamental cutset matrix Q_f:

$$Q_f = \left[\begin{array}{c|c} B^T_{f_{12}} & U \end{array} \right]$$

where $B^T_{f_{12}}$ denotes the transpose of $B_{f_{12}}$.

Step 5: Form an incidence matrix A:

$$A = D \times Q_f$$

where D is nonsingular ([1], [7]). This corresponds to the application of ring-sum operations on the rows of Q_f until there are no more than two 1's in each column of the transformed matrix A. If the order of Q_f is large, apply Mayeda's method [8] to transform Q_f into A.

Step 6: Form the incidence matrix A_a by adding a last row to A so that each column of A_a has exactly two 1's.

Step 7: Determine the network N* from A_a and then remove the edge identified by x_o. This gives the desired network N which realizes the given $q_{i,j}$ reliability as its function, and the two terminals i and j of N are the endpoints of the removed edge x_o.

Example 1: Suppose that a given terminal reliability function $q_{i,j}$ has its canonical terms represented by the table of combination in Table I, in which each row represents a canonical term. An entry "1" under a variable designates that the variable is unprimed and the zeros designate otherwise. Thus the first row means $p'_1\ p_2\ p'_3\ p_4\ p_5$ thus giving a path set (2, 4, 5). Checking the combinations in Table I, we find that the canonical terms are conformable, thus we can start on the path sets, which are

$$S_1 : (1) \ ; \ S_2 : (2, 3) \ ; \ S_3 : (2, 4, 5)$$

The path matrix P between vertices i and j is

$$P = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

The matrix B_p is given by

$$B_p = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_0 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right] \end{array}$$

Adding row 2 to row 3 in B_p, the matrix B_1 is obtained.

TABLE I

P_1	P_2	P_3	P_4	P_5
0	1	0	1	1
0	1	1	0	0
0	1	1	0	1
0	1	1	1	0
0	1	1	1	1
1	0	0	0	0
1	0	0	0	1
1	0	0	1	0
1	0	0	1	1
1	0	1	0	0
1	0	1	0	1
1	0	1	1	0
1	0	1	1	1
1	1	0	0	0
1	1	0	0	1
1	1	0	1	0
1	1	0	1	1
1	1	1	0	0
1	1	1	0	1
1	1	1	1	0
1	1	1	1	1

$$B_1 = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_0 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \end{array}$$

Now adding row 3 to row 2 in B_1, we find

$$B_f = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_0 \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \end{array} = \left[\begin{array}{c|c} U & B_{f_{12}} \end{array}\right]$$

which is a fundamental circuit matrix with x_4, x_5, x_0 as the branches of the three, and x_1, x_2, x_3 as chords.

Thus,

$$Q_f = \left[B^T_{f_{12}} \;\vdots\; U \right] = \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Hence,

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$A_a = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_0 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

The graph G* which includes the added element x_o is shown in Figure 1.

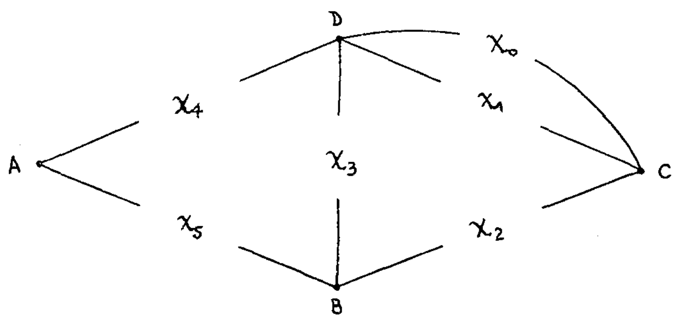

Fig. 1. Graph G* (or N*).

Finally, with x_o removed from G*, the desired graph G with terminals i and j identified by the two endpoints of x_o is obtained and shown in Figure 2.

Notice that as long as a collection of e - v + 2 independent path sets is contained in a function, we can find a graph G to provide all the communication paths specified by the function. However, the realization may provide more paths than that specified although it might be considered a minimal realization, since the method used here utilizes the minimum number of branches (exactly the same number the given function calls for). In the light of these facts, the following convention is useful.

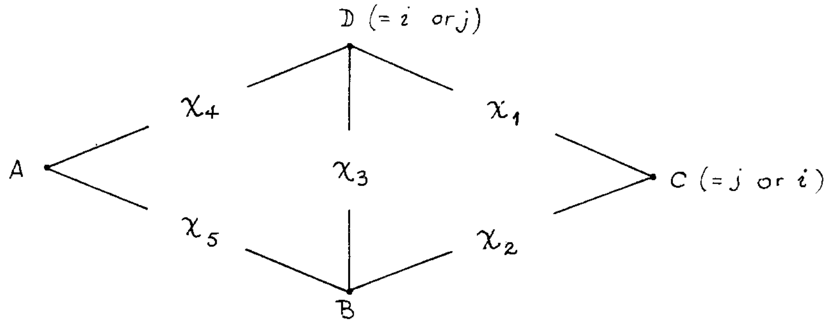

Fig. 2. Graph G (or N).

Convention: If a graph G can be constructed to provide the communication requirements specified by a given function and with the same number of elements as the number of variables contained in the function, then the function is considered realizable by G. If G provides only the specified requirements of communication, the realization is considered to be "exact". If G is the only possible realization, the synthesis is "unique".

As illustrated by Example 1, the major steps of synthesizing a $q_{i,j}$ function after checking for conformability of the canonical terms are: to obtain the circuit matrix B_p from which B_f is derived; then to get Q_f; and, after chacking realizability of Q_f, to construct a graph from Q_f. There is no problem of getting a matrix Q_f from a given function [8]. Whether this matrix is a fundamental cut-set matrix of a graph depends on whether this matrix can pass Mayeda's realizability test. Thus we have the following theorem.

Theorem 3: A terminal reliability function, $q_{i,j}$, is realizable exactly if and only if

(1) the canonical terms of $q_{i,j}$ are conformable;
(2) the matrix Q_f derived from $q_{i,j}$ is a fundamental cut-set matrix.

Proof of Sufficiency: The second condition implies that Q_f is realizable as a fundamental cut-set matrix of a graph G [8]. Since Q_f is derived from B_f, both Q_f and B_f correspond to the same fundamental system [1] of a tree in G. Since B_f is obtained from the path sets in $P_{i,j}$ of the given $q_{i,j}$ function, the graph G contains a pair of terminals i and j, between which the path sets in $P_{i,j}$ can be found. With the first condition all terms in $q_{i,j}$ are guaranteed in the graph G. Thus the conditions are sufficient.

Proof of Necessity: If Q_f derived from $q_{i,j}$ is not realizable as a fundamental cut-set matrix of any graph, its corresponding B_f is not a fundamental circuit matrix of any fundamental system with respect to any chosen complete tree of any graph. Thus, there is no graph, which possesses a pair of terminal vertices, between which the complete collection of path sets $P_{i,j}$ corresponds exactly to that of the given $q_{i,j}$ function. Now even if the second condition is satisfied, all terms in $q_{i,j}$ must be conformable, otherwise the realization is not exact.

5. *Synthesis of partially specified functions*

As will be shown later, unique realization is guaranteed when v-1 independent but related terminal reliability functions are given. When one (or more) of these functions is specified, realizations which are unique within a 2-isomorphism are possible provided that the expression of the function contains all its canonical terms. If only some of the path terms of a given reliability function are given, the realization cannot be unique or even exact. However, the realization will be minimal and will provide the necessary communiaction paths with more than the required reliability.

As an illustration, suppose that a terminal reliability function is given by

$$q_{i,j} = p_1 p_2' p_3 p_4 p_5' + p_1' p_2 p_3 p_4 p_5$$

where $p_1 = 0.9$, $p_2 = 0.1$, $p_3 = 0.9$, $p_4 = 0.8$, and $p_5 = 0.2$.

It is not possible to realize this function exactly. But using the specified branch reliabilities realizations giving at least the specified value of $q_{i,j}$ (i.e. 0.46800) can easily be obtained. Two such realizations, G_a and G_b are shown in Figure 3. Graph G_a gives a reliability value of 0.65304,

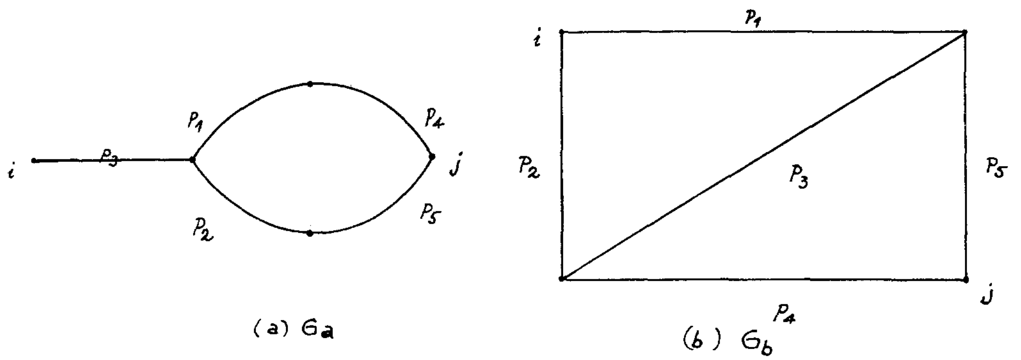

Fig. 3.

which is higher than that specified. This is due to the added reliability given by the additional secondary path terms.

Both graphs in Figure 3 contain the circuit set (1, 2, 4, 5), and edge 3 is common to both path sets, (1, 3, 4) and (2, 3, 5). However, it is obvious that graph G_b will give higher reliability as it provides more communication paths, (2,4) and (1,5), in addition to those in graph G_a.

6. *Synthesis of related $q_{i,j}$ functions*

The synthesis of a single $q_{i,j}$ function empasizes only two vertices in the realization. When two or more $q_{i,j}$ functions are to be synthesized as a related group, not only each function must be realizable, but they must also be realized by the same graph. That is, we must be able to obtain all the given functions from the same realization.

Two terminal reliability functions are said to be *compatible* with one another if and only if there exists a graph, from which these functions can be obtained.

Suppose that two terminal reliability functions are given. Ruling out the trivial case of two equivalent functions, there exist two possibilities. The first is that the two given functions associate with only three terminal vertices, e.g., they are $q_{i,j}$ and $q_{j,k}$. The second is that they associate with four distinct terminal vertices, e.g., $q_{i,j}$ and $q_{r,t}$.

Theorem 4: Compatible terminal reliability functions possess an identical complete collection of circuit sets.

Proof: Suppose that among the functions there exist two or more different complete collections of circuit sets. This means that the individual realizations of the functions are not all 2-isomorphic. Thus there exist no graph to which all these functions belong. This contradicts the definition of compatibility. Hence the theorem is true.

Corollary: An identical fundamental circuit matrix (and, thus, fundamental cut-set matrix) can be derived from compatible terminal reliability functions.

Theorem 4 or its corollary gives us a strong necessary condition for the compatibility of terminal reliability functions.

Theorem 5: Let $q_{i,j}$ and $q_{j,k}$ be two realizable functions. Then $q_{i,j}$ and $q_{j,k}$ are compatible if and only if their corresponding complete collections of circuit sets are identical.

Proof: If $q_{i,j}$ and $q_{j,k}$ are compatible, then they correspond to the same graph G and hence generate the same complete collection of circuit sets. The sufficiency of the theorem follows from the fact that the realization of a circuit matrix is unique to within a 2-isomorphism [1].

The condition in Theorem 5 guarantees not only the compatibility of the two functions, $q_{i,j}$ and $q_{j,k}$ but also the generated function $q_{i,k}$. In fact, if $q_{i,k}$ were not compatible with either of its generator functions, the generator functions could not be compatible with each other. Thus we have the following corollary.

Corollary: If two related functions $q_{i,j}$ and $q_{j,k}$ are compatible with each other, they are also compatible with their generated function $q_{i,k}$; and conversely.

The following examples illustrate the applications of the last two theorems in the synthesis of two related functions.

Example 2: Synthesize $q_{i,j}$ and $q_{j,k}$ of which the path sets are given as

$$P_{i,j} : (3,4)\ ;\ (1,4,6);\ (5,6,7);\ (1,3,5,7);\ (2,7)$$

$$P_{j,k} : (3,4,6);\ (1,4);\ (2,3,4,5);\ (5,7);\ (2,6,7)$$

From $P_{i,j}$, a complete collection of circuit sets can be obtained under ring sum operations as

$$B_{i,j}: (1,3,6);\ (3,4,5,6,7);\ (1,4,5,7);\ (2,3,4,7);\ (2,5,6);$$
$$(1,2,3,5);\ (1,2,4,6,7).$$

The ring sum of the second set in $P_{i,j}$ and the second set in $P_{j,k}$ is obviously a path set, $S_{i,k}$: (6), in $P_{i,k}$. With this set and the sets in $B_{i,j}$ we can obtain the following path sets with respect to vertices i and k:

$$P_{i,k}: (6);\ (1,3);\ (3,4,5,7);\ (2,5);\ (1,2,4,7).$$

A simple check shows that the collections of circuit sets $B_{j,k}$ and $B_{i,k}$ from $P_{j,k}$ and $P_{i,k}$ respectively, are identical with $B_{i,j}$ so that they are all equal to B of a graph, and thus, by Theorem 5 and its corollary $q_{i,k}$ corresponding to $P_{i,k}$, together with its generator functions $q_{i,j}$ and $q_{j,k}$, are compatible functions.

The circuit matrix using the circuit sets in $B_{i,j}$ as rows is of rank 3. Taking the three independent circuit sets, (3, 4, 5, 6, 7), (1, 4, 5, 7), and (2, 5, 6) from $B_{i,j}$ we form B_f:

$$B_f = \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \end{array} = \begin{bmatrix} U_3 & B_{f_{12}} \end{bmatrix}$$

Thus,

$$Q_f = \begin{bmatrix} B_{f_{12}}^T & U_4 \end{bmatrix} = \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \left[\begin{array}{ccc|cccc} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

The graph realizing Q_f is shown in Figure 4.

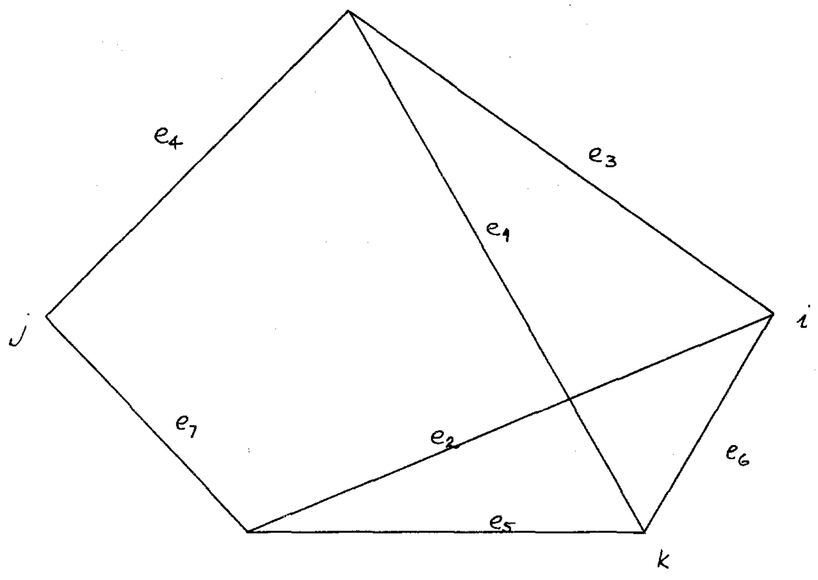

Fig. 4. Realization for Example 2.

Note that, in this example, one of the path sets contains only one element (e_6 across i and k) so that in the realization, we need not add a forcing element as was done in Example 1. Terminal j can be readily identified by inspection of any path from $P_{i,j}$ (or $P_{j,k}$).

Example 3: Let it be required to synthesize $q_{i,j}$ and $q_{j,k}$ of which the path sets are given as

$P_{i,j}$: (1, 5); (2, 4); (1, 3, 4); (2, 3, 5)

$P_{j,k}$: (1, 4); (2, 5); (1, 3, 5); (2, 3, 4).

The complete collections of circuit sets from both $P_{i,j}$ and $P_{j,k}$ are found to be the same and that is

B: (1, 2, 4, 5); (3, 4, 5); (1, 2, 3).

The ring sum of the first set in $P_{i,j}$ and $P_{j,k}$ gives $S_{i,k}$: (4, 5), which is a path set in $P_{i,k}$. With the sets in B, one obtains

$$P_{i,k} : (4,5);\ (1,2);\ (1,2,3).$$

Clearly the sets in $P_{i,k}$ cannot produce all the sets in B.

Although the discussions so far in this section have been concerned with two or three functions involving three related terminal vertices, no generality has been lost. That is to say, more than three terminal reliability functions associated with any number of similarly related terminal vertices can be tested for compatibility and synthesized in the same manner as previously discussed. However, there exists a limit to the number of functions which can be compatible as a group after one of them is specified. This will be shown in a later section.

7. *Synthesis of q_n*

Next, we consider simultaneously all the terminals of a given probabilistic network. The problem to be considered is the following: If a probabilistic network has infinite branch capacities, what is the possibility that every pair of terminals can communicate with each other? But first, a few new terms are defined.

Simultaneous communication is an event that every pair of terminals i and j ($i \neq j$) in a network can communicate with each other in a specified period of time. A *network reliability function* q_n is defined to be an expression of the probability of simultaneous communication in a network.

To obtain q_n from a given network, one can first determine a complete set of trees for the network. Simultaneous communication is possible if and only if the unprimed variables of a canonical term include those corresponding to the branches of a tree. Then, by taking the sum of all canonical terms, each of which contains a product of unprimed variables with their associated links forming a tree or tree plus chords, the function q_n is obtained. Also, the sum of all the canonical terms common to all $q_{i,j}$ functions of the network is q_n.

The canonical expression of q_n contains two types of terms: *a tree term* of q_n is a canonical term in which the elements associated with the unprimed variables correspond to all the branches of a tree; a *secondary tree term* of q_n is a canonical term in which the elements associated with the unprimed variables correspond to all the branches of a tree plus some chords.

The canonical terms of a reliability function q_n are said to be *conformable* if and only if the function contains all the secondary tree terms associated with every tree term of the function.

It is relatively simpler to recognize tree terms of q_n than path terms of $q_{i,j}$ because of the fact that all tree terms contain the same number of unprimed variables. In fact, they are terms with the least number of unprimed variables.

The synthesis of q_n will be accomplished if the corresponding collection of tree sets in q_n are realized by a graph. To synthesize a tree matrix, one can first derive a fundamental cut-set matrix from the tree matrix [9], and then synthesize the cut-set matrix as has been done previously. Thus the procedure of synthesizing a given network reliability function is rather straightforward.

8. *Uniqueness of realization*

If a reliability function can be obtained from two or more graphs which are not isomorphic, the realization of this function cannot be unique. The realization of a terminal reliability function may or may not be unique. However, a given terminal reliability function can possess one and only one complete collection of circuit sets, and its realizations must contain the same circuit sets even if they are not isomorphic. This leads to the following theorem.

Theorem 6: The realization of a terminal reliability function is unique to within a 2-isomorphism.

Since isomorphic graphs are also 2-isomorphic [1], the above theorem does not exclude the possibility of unique realization of a function $q_{i,j}$. Now, with Theorems 4, 5 and 6, the next theorem is obvious.

Theorem 7: The realization of a collection of compatible terminal reliability functions is unique to within a 2-isomorphism.

The conditions for unique synthesis of a collection of $q_{i,j}$ functions can now be investigated.

A graph G with v vertices will have $\frac{1}{2}$ v(v-1) different $q_{i,j}$ functions. With $q_{i,j}$ and $q_{j,k}$ given, the third related function $q_{i,k}$ is already specified tacitly and can easily be generated. In fact, Theorem 2 shows that there exist, at most, v-1 independent $q_{i,j}$ functions in G. This leads to the following uniqueness theorem.

Theorem 8: The realization of a set of n = v-1 independent and compatible terminal reliability functions is unique.

Proof: If v-1 independent $q_{i,j}$ functions are given, all $\frac{1}{2}$ v(v-1) related functions can be obtained. This means that by inspection of the canonical terms of the functions, every element can be identified as a "direct-path" element with respect to its incident pair of vertices. Thus the one and only one incidence matrix from this group of functions can be obtained, and the graph corresponding to this matrix is the only realization of the functions so that it is unique.

9. *Conclusions*

In this paper, some of the properties of terminal reliability functions have been discussed and the problem of realization of reliability functions of a probabilistic communication network with distinct branch reliabilities based on these properties have been studied. Synthesis procedures for one or more prescribed terminal reliability functions as well as network reliability functions have been developed and subsequently illustrated by the examples. Also, the problem of uniqueness of realization has been discussed and has led to the result of a uniqueness theorem (Theorem 8).

It is worth noting that for a relatively complicated function, the process of checking for conformability of canonical terms and then determining the path terms is tedious although straightforward. However, such a difficulty can be overcome by the use of digital computation.

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